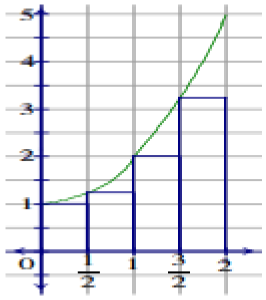
Unit 6 – Integration and Accumulation of Change

Approximating Areas with Riemann Sums:

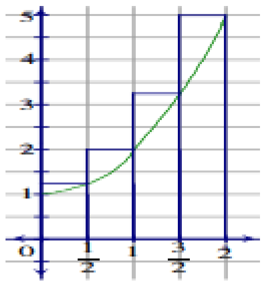
a) **Left edge Rectangles** $f(x)=x^2+1$ from $[\,0,2]$ using 4 subdivisions

(Find area of each rectangle and add together)



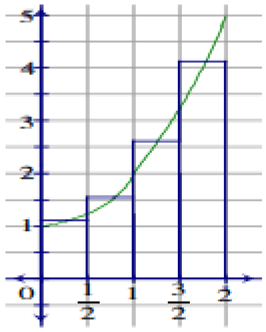
b) **Right edge Rectangles** $f(x)=x^2+1$ from $[\,0,2]$ using 4 subdivisions

(Find area of each rectangle and add together)



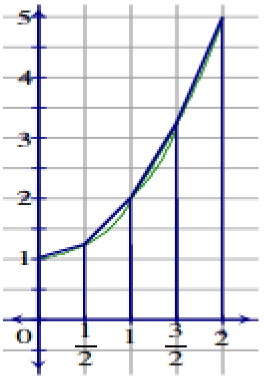
c) **Midpoint Rectangles** $f(x)=x^2+1$ from $[\,0,2]$ using 4 subdivisions

(Find area of each rectangle and add together)



d) ***Trapezoidal Rule**

$Area \approx$



Actual Area $=\int_0^2 \left(x^2+1\right)dx \;=$

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***Approximating Area when given data only (no equation given)**

To estimate the area of a plot of land, a surveyor takes several measurements. The measurements are taken every 15 feet for the 120 ft. long plot of land, where y represents the distance across the land at each 15 ft. increment.

x	0	15	30	45	60	75	90	105	120
y	58	63	72	60	62	69	61	74	67

a) Estimate using Trapezoidal Rule

$A \doteq$

b) Estimate using 4 Midpoint subdivisions

$A \doteq$

c) Estimate Avg. value using Trapezoidal Rule

Avg. Value $\doteq$

d) What are you finding in part c?

e) Estimate using Left Endpoint

$A \doteq$

f) Estimate using Right Endpoint

$A \doteq$

Summation Notation and Definite Integral Notation:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i)$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + \Delta x \cdot i$.

EX#1: Fill in the following spaces for the definite integral written in summation notation.

$$\int_2^6 \frac{1}{5}x^2\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\quad + \quad \right) \cdot \frac{}{}$$

EX#2: Fill in the following spaces for the definite integral written in summation notation.

$$\int_\pi^{2\pi} \cos(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{}{} \cdot \cos\left(\quad + \frac{}{} \right)$$

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Integration Formulas

***Integral of a constant** $\int a\,dx =$ **EX#1:** $\int 5\,dx =$ **EX#2:** $\int \pi\,dx =$

***Polynomials** $\int x^n\,dx =$ **EX#1:** $\int x^3 + 5x^2 - 8x\,dx =$

***Fractions** (Bring up denominator, then take integral)

EX#1: $\int \frac{1}{x^4}\,dx \Rightarrow \int x^{-4}\,dx =$

***Constant** $^{\text{Variable}}$ $\int a^x\,dx = \frac{a^x}{1 \cdot \ln a} + C$ **(3 steps : itself, divided by deriv. of exponent, divided by ln of base)**

EX#1: $\int 5^x\,dx =$ **EX#2:** $\int 3^{2x}\,dx =$

EX#3: $\int e^x\,dx =$ **EX#4:** $\int e^{2x}\,dx =$

***Trig Functions** **(Always divide by derivative of the angle)**

$\int \sin x\,dx =$ $\int \cos x\,dx =$

EX#1: $\int \cos 2x\,dx =$ **EX#2:** $\int \sin 6x\,dx =$

***Natural Log:** $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$ $\Rightarrow$ top is the derivative of the bottom

EX#1: $\int \frac{1}{x}\,dx =$ **EX#2:** $\int \frac{3x^2}{x^3 - 5}\,dx =$

EX#3: $\int \frac{-\sin x}{\cos x}\,dx =$ **EX#4:** $\int \frac{x^3}{x^4 + 1}\,dx =$

The Fundamental Theorem of Calculus:

2nd Fundamental Theorem of Calculus (When taking the derivative of an integral)

Plug in the variable on top times its derivative minus plug in the variable on bottom times its derivative.

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

EX#1: $\frac{d}{dx}\int_0^x t^3\,dt =$ **EX#2:** $\frac{d}{dx}\int_{x^2}^0 t^3\,dt =$

EX#3: $\frac{d}{dx}\int_x^{x^2} \sqrt{t^5 + 2}\,dt =$

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Definite Integrals:

1st Fundamental Theorem of Calculus

Just plug in the top # minus the bottom #.

EX#1: $\int_0^2 x^2\,dx =$

EX#2: $\int_{\pi/4}^{\pi} \sin x\,dx =$

Integrating Using u-Substitution:

***Substitution** When integrating we usually let u = the part in the parenthesis, the part under the radical, the denominator, the exponent, or the angle of the trig. function.

EX#1: $\int x\sqrt{x^2+1}\,dx$

$u =$

$du =$

EX#2: $\int \frac{x^2}{(2x^3+5)^4}\,dx = \int x^2(2x^3+5)^{-4}\,dx =$

$u =$ $du =$

EX#3: $\int x\sqrt{x+1}\,dx =$

$u =$ $x =$

$du =$

EX#4: $\int x\cos x^2\,dx =$

$u =$

$du =$

EX#5: $\int_0^1 2x^2(2x^3+1)^4\,dx =$

$u =$ You must switch everything from x to u. Including the #'s.

$du =$

Integrating Functions Using Long Division and Completing the Square:

***Integral (top is higher or same power than bottom)** **(Must divide bottom equation into top equation).**

EX#1: $\int \frac{x^2+2}{x^2-2x+4}\,dx \Rightarrow$ **Long Division** $=$

EX#2: $\int \frac{x^2+3x-5}{x}\,dx =$

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*Inverse Trig Functions

$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx =$ $\int \frac{1}{a^2 + x^2}\,dx =$ $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx =$

Find variable v and constant a. The top MUST be the derivative of the variable v.

EX#1: $\int \frac{1}{\sqrt{9 - x^2}}\,dx =$

$v =$ $a =$

EX#4: $\int \frac{1}{\sqrt{4 - 9x^2}}\,dx =$

$v =$ $a =$

EX#2: $\int \frac{1}{16 + x^2}\,dx =$

$v =$ $a =$

EX#5: $\int \frac{1}{9x^2 + 16}\,dx =$

$v =$ $a =$

EX#3: $\int \frac{1}{x\sqrt{x^2 - 25}}\,dx =$

$v =$ $a =$

EX#6: $\int \frac{6}{x\sqrt{49x^2 - 25}}\,dx =$

$v =$ $a =$

Sample AP Problems:

2013 AP Practice Exam Multiple Choice

3. Which of the following definite integrals has the same value as $\int_0^4 xe^{x^2}\,dx$?

(A) $\frac{1}{2}\int_0^4 e^u\,du$

(B) $\frac{1}{2}\int_0^{16} e^u\,du$

(C) $2\int_0^2 e^u\,du$

(D) $2\int_0^4 e^u\,du$

(E) $2\int_0^{16} e^u\,du$

6. $\int_2^4 \frac{dx}{5 - 3x} =$

(A) $-\ln 7$ (B) $-\frac{\ln 7}{3}$ (C) $\frac{\ln 7}{3}$ (D) $\ln 7$ (E) $3\ln 7$

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20. For $x > 0$, $\frac{d}{dx}\left(\int_0^{2x} \ln\left(t^3+1\right) dt\right) =$

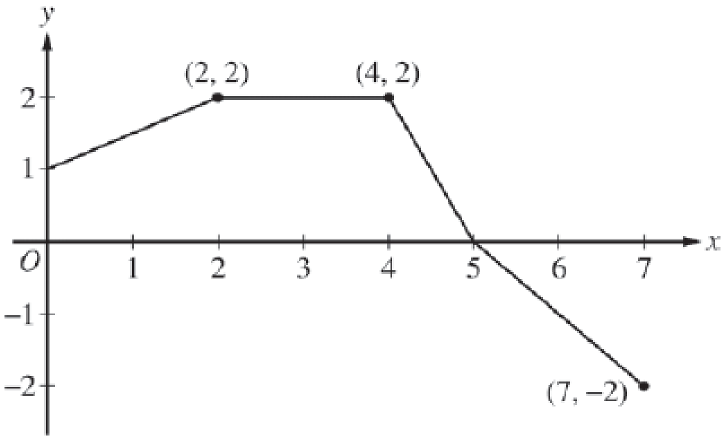
(A) $\ln\left(x^3+1\right)$

(B) $\ln\left(8x^3+1\right)$

(C) $2\ln\left(x^3+1\right)$

(D) $2\ln\left(8x^3+1\right)$

(E) $24x^2\ln\left(8x^3+1\right)$



Graph of f

21. The graph of a function f is shown above. What is the value of $\int_0^7 f(x)\, dx$?

(A) 6 (B) 8 (C) 10 (D) 14 (E) 18

26. Which of the following is an antiderivative of $f(x) = \sqrt{1+x^3}$?

(A) $\frac{2}{3}\left(1+x^3\right)^{3/2}$

(B) $\dfrac{\frac{2}{3}\left(1+x^3\right)^{3/2}}{3x^2}$

(C) $\int_0^{1+x^3} \sqrt{t}\, dt$

(D) $\int_0^{x^3} \sqrt{1+t}\, dt$

(E) $\int_0^{x} \sqrt{1+t^3}\, dt$

78. Let f and g be continuous functions such that $\int_0^{10} f(x)\, dx = 21$, $\int_0^{10} \frac{1}{2}g(x)\, dx = 8$, and $\int_3^{10} (f(x) - g(x))\, dx = 2$. What is the value of $\int_0^{3} (f(x) - g(x))\, dx$?

(A) 3 (B) 7 (C) 11 (D) 15 (E) 19

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x	0	0.5	1	1.5	2	2.5	3
$f(x)$	0	4	10	18	28	40	54

83. The table above gives selected values for a continuous function f. If f is increasing over the closed interval $[0,3]$, which of the following could be the value of $\int_0^3 f(x)\,dx$?

(A) 50 (B) 62 (C) 77 (D) 100 (E) 154

91. The function f is continuous and $\int_0^8 f(u)\,du = 6$. What is the value of $\int_1^3 xf\left(x^2-1\right)dx$?

(A) $\frac{3}{2}$ (B) 3 (C) 6 (D) 12 (E) 24

2014 AP Practice Exam Multiple Choice

1. $\int_2^x \left(3t^2-1\right)dt =$

(A) $x^3 - x - 6$ (B) $x^3 - x$ (C) $3x^2 - 12$ (D) $3x^2 - 1$ (E) $6x - 12$

4. $\int_1^2 \frac{dx}{2x+1} =$

(A) $2\ln 2$ (B) $\frac{1}{2}\ln 2$ (C) $2(\ln 5 - \ln 3)$ (D) $\ln 5 - \ln 3$ (E) $\frac{1}{2}(\ln 5 - \ln 3)$

8. Using the substitution $u = \sin(2x)$, $\int_{\pi/6}^{\pi/2} \sin^5(2x)\cos(2x)\,dx$ is equivalent to

(A) $-2\int_{1/2}^{1} u^5\,du$

(B) $\frac{1}{2}\int_{1/2}^{1} u^5\,du$

(C) $\frac{1}{2}\int_0^{\sqrt{3}/2} u^5\,du$

(D) $\frac{1}{2}\int_{\sqrt{3}/2}^{0} u^5\,du$

(E) $2\int_{\sqrt{3}/2}^{0} u^5\,du$

12. Let f be the function given by $f(x) = 9^x$. If four subintervals of equal length are used, what is the value of the right Riemann sum approximation for $\int_0^2 f(x)\,dx$?

(A) 20 (B) 40 (C) 60 (D) 80 (E) 120

26. For $x > 0$, $\frac{d}{dx}\int_1^{\sqrt{x}} \frac{1}{1+t^2}\,dt =$

(A) $\frac{1}{2\sqrt{x}(1+x)}$ (B) $\frac{1}{2\sqrt{x}\left(1+\sqrt{x}\right)}$ (C) $\frac{1}{1+x}$ (D) $\frac{\sqrt{x}}{1+x}$ (E) $\frac{1}{1+\sqrt{x}}$

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77. If $\sin\left(\frac{1}{x^2+1}\right)$ is an antiderivative for $f(x)$, then $\int_1^2 f(x)\,dx =$

(A) -0.281 (B) -0.102 (C) 0.102 (D) 0.260 (E) 0.282

83. Let f and g be continuous functions such that $\int_0^6 f(x)\,dx = 9$, $\int_3^6 f(x)\,dx = 5$, and $\int_3^0 g(x)\,dx = -7$. What is the value of $\int_0^3 \left(\frac{1}{2}f(x) - 3g(x)\right) dx$?

(A) -23 (B) -19 (C) $-\frac{17}{2}$ (D) 19 (E) 23

85. A particle moves along the x-axis so that its velocity at time $t \ge 0$ is given by $v(t) = \frac{t^2-1}{t^2+1}$. What is the total distance traveled by the particle from $t = 0$ to $t = 2$?

(A) 0.214 (B) 0.320 (C) 0.600 (D) 0.927 (E) 1.600

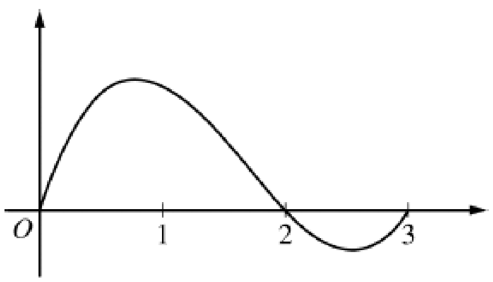
87. A differentiable function f has the property that $f'(x) \le 3$ for $1 \le x \le 8$ and $f(5) = 6$. Which of the following could be true?

I. $f(2) = 0$

II. $f(6) = -2$

III. $f(7) = 13$

(A) I only
(B) II only
(C) I and II only
(D) I and III only
(E) II and III only



Graph of f

88. The graph of the differentiable function f is shown in the figure above. Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Which of the following correctly orders $h(2)$, $h'(2)$, and $h''(2)$?

(A) $h(2) < h'(2) < h''(2)$

(B) $h'(2) < h(2) < h''(2)$

(C) $h'(2) < h''(2) < h(2)$

(D) $h''(2) < h(2) < h'(2)$

(E) $h''(2) < h'(2) < h(2)$