Unit 2 - Differentiation: Definition and Fundamental Properties

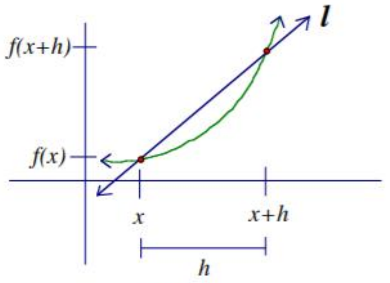
Definition of a Derivative:

Derivative at all points

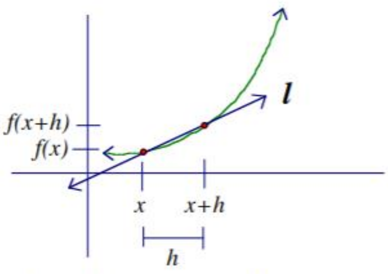
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative at the point $(a, f(a))$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Line l is a secant line



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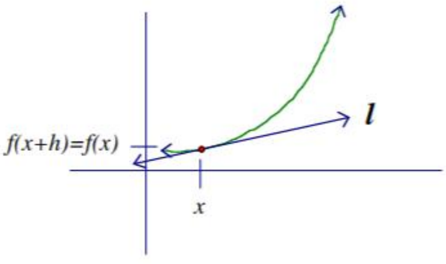

Line l is a tangent line

slope of secant line $l = \dfrac{f(x+h) - f(x)}{x+h-x}$

$\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ means that the distance h is approaching 0 and the points get closer to each other and the two points become the same point and line l is now a tangent line.

The derivative of a function finds the slope of the tangent line!

EX #1 :

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} \quad =$$

EX #2 :

$$f'(x) = \lim_{h \to 0} \frac{4(2+h)^3 - 32}{h} \quad =$$

** The derivative finds the __________ to the tangent line.

** The normal line is ____________________ to the tangent line.

EX #3 : $f(x) = 5x^2$ Find equation of the tangent line and normal line at $x = 3$.

Equation of a Line (point - slope form) : $y - y_1 = m(x - x_1)$

Equation of the tangent line :

Equation of the normal line :

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When you see these problems, you need to take a derivative of the given equation.

EX#1: $\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} =$

EX#2: $\lim_{h \to 0} \frac{3(x+h)^4 - 3x^4}{h} =$

EX#3: $\lim_{h \to 0} \frac{5(2+h)^3 - 40}{h} =$

EX#4: $\lim_{h \to 0} \frac{(1+h)^4 - 1}{h} =$

Derivative Formulas:

***Power Rule** $y = x^n$ $y' =$

EX#1: $y = 2x^5$ $y' =$

EX#2: $y = \frac{5}{x}$ $y' =$

***Product Rule** $y = f(x) \cdot g(x)$ $y' =$

EX #1: $y = x^2 \sin x$ $y' =$

***Quotient Rule** $y = \frac{f(x)}{g(x)}$ $y' =$

EX #1: $y = \frac{\sin x}{x^3}$ $y' =$

***Chain Rule** $y = (f(x))^n$ **OR** $y = f(g(x))$

$y' =$

EX #1: $y = (x^2 + 1)^3$ $y' =$

***Trig. Functions** (Take the derivative of the trig. function times the derivative of the angle)

Function	*Derivative*	*Example*
$\sin x$		$\frac{d}{dx}\sin(x^2) =$
$\cos x$		$\frac{d}{dx}\cos^2(3x^3) =$
$\tan x$		$\frac{d}{dx}\tan(25x) =$
$\csc x$		$\frac{d}{dx}\csc(3x^4) =$
$\sec x$		$\frac{d}{dx}\sec(\sin x) =$
$\cot x$		$\frac{d}{dx}\cot(x^5) =$

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***Natural Log** $y = ln(f(x))$ $y' =$

EX#1: $y = ln(x^2+1)$ $y' =$

EX#2: $y = ln(\sin x)$ $y' =$

EX#3: $y = log\, x^2$ change of base $\Rightarrow$ $y = \dfrac{ln\, x^2}{ln\, 10}$ $y' =$

***Constant$^{\text{Variable}}$** $y = a^{f(x)}$ $y' =$

(3 steps: itself, derivative of exponent, ln of base)

EX#1: $y = 2^x$
$y' =$

EX#2: $y = 3^{x^2}$
$y' =$

EX#3: $y = e^{5x}$
$y' =$

Sample AP Problems:

2013 AP Practice Exam Multiple Choice

2. If $f(x) = x^3 - x^2 + x - 1$, then $f'(2) =$

(A) 10 (B) 9 (C) 7 (D) 5 (E) 3

7. Let f be the function given by $f(x) = x^3 - 6x^2 + 8x - 2$. What is the instantaneous rate of change of f at $x = 3$?

(A) -5 (B) $-\frac{15}{4}$ (C) -1 (D) 6 (E) 17

$$f(x) = \begin{cases} x + b & \text{if } x \le 1 \\ ax^2 & \text{if } x > 1 \end{cases}$$

14. Let f be the function given above. What are all values of a and b for which f is differentiable at $x = 1$?

(A) $a = \frac{1}{2}$ and $b = -\frac{1}{2}$

(B) $a = \frac{1}{2}$ and $b = \frac{3}{2}$

(C) $a = \frac{1}{2}$ and b is any real number

(D) $a = b + 1$, where b is any real number

(E) There are no such values of a and b.

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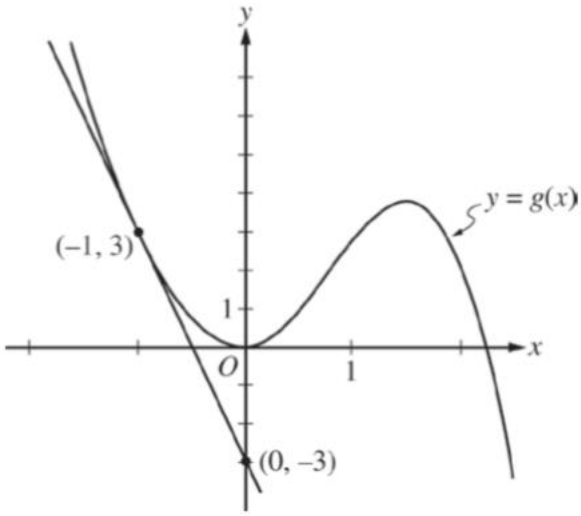
$f(3)$	$g(3)$	$f'(3)$	$g'(3)$
–1	2	5	–2

15. The table above gives values for the functions f and g and their derivatives at $x = 3$. Let k be the function given by $k(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$. What is the value of $k'(3)$?

(A) $-\frac{5}{2}$ (B) -2 (C) 2 (D) 3 (E) 8

16. If $y = 5x\sqrt{x^2+1}$, then $\frac{dy}{dx}$ at $x = 3$ is

(A) $\frac{5}{2\sqrt{10}}$ (B) $\frac{15}{\sqrt{10}}$ (C) $\frac{15}{2\sqrt{10}} + 5\sqrt{10}$ (D) $\frac{45}{\sqrt{10}} + 5\sqrt{10}$ (E) $\frac{45}{\sqrt{10}} + 15\sqrt{10}$



19. The figure above shows the graph of the function g and the line tangent to the graph of g at $x = -1$. Let h be the function given by $h(x) = e^x \cdot g(x)$. What is the value of $h'(-1)$?

(A) $\frac{9}{e}$ (B) $\frac{-3}{e}$ (C) $\frac{-6}{e}$ (D) $\frac{-6}{e} - \frac{3}{e^2}$ (E) -6

2014 AP Practice Exam Multiple Choice

2. What is the slope of the line tangent to the graph of $y = \ln(2x)$ at the point where $x = 4$?

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$ (E) 4

3. If $f(x) = 4x^{-2} + \frac{1}{4}x^2 + 4$, then $f'(2) =$

(A) -62 (B) -58 (C) -3 (D) 0 (E) 1

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6. $\frac{d}{dx}\left(\sin^3\left(x^2\right)\right) =$

(A) $\cos^3\left(x^2\right)$

(B) $3\sin^2\left(x^2\right)$

(C) $6x\sin^2\left(x^2\right)$

(D) $3\sin^2\left(x^2\right)\cos\left(x^2\right)$

(E) $6x\sin^2\left(x^2\right)\cos\left(x^2\right)$

13. $\frac{d}{dx}\left(\frac{x+1}{x^2+1}\right) =$

(A) $\frac{x^2+2x-1}{\left(x^2+1\right)^2}$

(B) $\frac{-x^2-2x+1}{x^2+1}$

(C) $\frac{-x^2-2x+1}{\left(x^2+1\right)^2}$

(D) $\frac{3x^2+2x+1}{\left(x^2+1\right)^2}$

(E) $\frac{1}{2x}$

17. If $f(x) = ae^{-ax}$ for $a > 0$, then $f'(x) =$

(A) e^{-ax}

(B) ae^{-ax}

(C) a^2e^{-ax}

(D) $-ae^{-ax}$

(E) $-a^2e^{-ax}$

20. $\lim_{x\to 2}\frac{\ln(x+3)-\ln(5)}{x-2}$ is

(A) 0 (B) $\frac{1}{5}$ (C) $\frac{1}{2}$ (D) 1 (E) nonexistent

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$$f(x) = \begin{cases} 3x + 5 & \text{when } x < -1 \\ -x^2 + 3 & \text{when } x \geq -1 \end{cases}$$

23. If f is the function defined above, then $f'(-1)$ is

(A) -3 (B) -2 (C) 2 (D) 3 (E) nonexistent

86. Line ℓ is tangent to the graph of $y = e^x$ at the point $\left(k, e^k\right)$. What is the positive value of k for which the y-intercept of ℓ is $\frac{1}{2}$?

(A) 0.405

(B) 0.768

(C) 1.500

(D) 1.560

(E) There is no such value of k.