When is a function continuous?

Recall that in order for a function *f* to be continuous at a point $(c, f(c))$, it must be the case that $\lim_{x \to c} f(x) = f(c)$. This is the definition of continuity at a point, often referred to as the "three-part definition" of continuity:

A function *f* is continuous at $x = c$ provided that

(i) $f(c)$ exists,

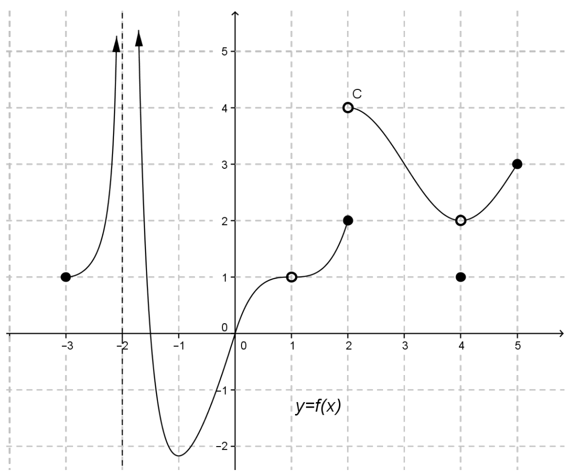
(ii) $\lim_{x \to c} f(x)$ exists, and

(iii) $\lim_{x \to c} f(x) = f(c)$.

Essentially, three quantities must all be the same finite number:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

All of the questions in this activity have to do with the function *f*, the graph shown here:



1. Complete the table below:

$x = c$	$\lim_{x \to c^-} f(x)$	$\lim_{x \to c^+} f(x)$	$f(c)$
$x = -2$			
$x = 1$			
$x = 2$			
$x = 4$			

2. The function $f(x)$ is discontinuous at each of the values of $x = c$ in the table in question 1. Use the information from that table to indicate the reason or reasons for the discontinuity at each $x = c$ by placing a check mark in every column that applies in the following table. [Note that a check mark for either Column 1 or Column 2 will necessarily result in a check mark for Column 3, because if one of the values is not a finite number, then it is not possible for another value to equal that value.]

	Column 1	**Column 2**	**Column 3**
$x = c$	$\lim_{x \to c} f(x)$ **is not a finite number**	$f(c)$ **is not a finite number**	$\lim_{x \to c} f(x) \neq f(c)$
$x = -2$			
$x = 1$			
$x = 2$			
$x = 4$			

The Intermediate Value Theorem

Recall that the **Intermediate Value Theorem (IVT)** states that if $f(x)$ is continuous on the interval $[a, b]$ with $f(a) \neq f(b)$, then if d is any number between $f(a)$ and $f(b)$, there is at least one c between a and b such that $f(c) = d$. More simply put, provided f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then every number between the output at a and the output at b is also an output for f.

So, for example, suppose that for the function $f(x)$, it is known that $f(0) = 3$ and $f(2) = 6$. We cannot use the IVT to support the claim that $f(c) = 5$ for some c between 0 and 2. Even though 5 is between 3 and 6, the IVT does not apply because we don't know if $f(x)$ is continuous.

In each of the following, decide whether the Intermediate Value Theorem (IVT) can or cannot be used to justify the stated conclusion. Then explain why or why not.

1. The following table gives some function values for the continuous function $g(x)$.

x	0	2	4
$g(x)$	7	3	-6

Therefore, there exists some r in the interval (0, 2) for which $f(r) = 5$.

☐ IVT applies

☐ IVT does not apply

Why or why not?

2. Let k be the function defined by $k(x) = \begin{cases} x+2, & x \leq 1 \\ x+4, & x > 1 \end{cases}$. There must exist some x between 0 and 2 for which $k(x) = 4$.

☐ IVT applies

☐ IVT does not apply

Why or why not?

3. Consider the function $s(x) = 2x^2 + 4x$. There must be some a between 0 and 1 for which $s(a) = 3$.

☐ IVT applies
☐ IVT does not apply

Why or why not?

4. Let $v(x) = \begin{cases} 2x - 1, & x < 2 \\ 1 - x, & x \geq 2 \end{cases}$. Then $v(x) = 1$ for some x in the interval $(0,\ 4)$.

☐ IVT applies
☐ IVT does not apply

Why or why not?

5. Which of the following statements follows from the statement: "If f is continuous on the interval $[1,\ 4]$ with $f(1) = 3$ and $f(4) = 9$, then there exists at least one value of c in the interval $(1,\ 4)$ for which $f(c) = 7$"?

a) If f is not continuous on $[1,\ 4]$ with $f(1) = 3$ and $f(4) = 9$, then there is no value of c in the interval $(1,\ 4)$ for which $f(c) = 7$.

b) If f is defined on the interval $[1,\ 4]$ with $f(1) = 3$ and $f(4) = 9$, and if there is at least one value of c in the interval $(1, 4)$ for which $f(c) = 7$, then f is continuous.

Does EVT or IVT apply?

Recall also that the **Extreme Value Theorem (EVT)** states that if $f(x)$ is continuous on the interval $[a, b]$, then $f(x)$ has both an absolute maximum value and an absolute minimum value on $[a, b]$.

A table is shown with selected function values for the twice differentiable function *k*. Read each of the explanations that follow and decide whether the Intermediate Value Theorem or the Extreme Value Theorem applies. Fill in the blanks with **IVT** or **EVT**, as appropriate.

x	1	2	3	4	5	6	7
$k(x)$	5	2	-4	-1	3	2	0

1. Since *k* is differentiable, it is also continuous. Since $k(6) = 2$ and $k(7) = 0$, and since 1 is between 2 and 0, it follows by the ______ that $k(c) = 1$ for some *c* between 6 and 7.

2. There must be a minimum value for *k* at some *r* in $[1, 7]$, because *k* is differentiable and therefore also continuous. Hence the ______ applies.

3. There must be some value *a* in $(2, 6)$ for which $f'(a) = 0$, because *k* is continuous and therefore *k* has an absolute maximum on $[0, 6]$ by the ______.

4. Since *k* is twice differentiable, then k' is differentiable – and therefore also continuous – it follows by the ______ applied to k' that $k'(c) = 0$ for some *c* in (a, b), and therefore in $(4, 6)$.

5. Which of the following statements follows from the statement: "If *f* is continuous on the interval $[a, b]$, then *f* has an absolute maximum and an absolute minimum on $[a, b]$"?

a) If *f* has an absolute maximum and an absolute minimum on $[a, b]$, then *f* is continuous on $[a, b]$.

b) If *f* has an absolute maximum but no absolute minimum on $[a, b]$, then *f* is not continuous on $[a, b]$.

c) Every function with an absolute maximum on $[a, b]$ also has an absolute minimum on $[a, b]$.

d) If *f* is not continuous on $[a, b]$, then *f* has no absolute maximum and no absolute minimum on $[a, b]$.